Bernoulli coding map and almost sure invariance principle for endomorphisms of $\mathbb{P}^k$

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Abstract

Let f be an holomorphic endomorphism of $\mathbb{P}^k$ and μ be its measure of maximal entropy. We prove an Almost Sure Invariance Principle for the systems $(\mathbb{P}^k, f, \mu)$. Our class $\mathcal{U}$ of observables includes the Hölder functions and unbounded ones which present analytic singularities. The proof is based on a geometric construction of a Bernoulli coding map $\omega : (\Sigma, s, \nu) \to (\mathbb{P}^k, f, \mu)$. We obtain the invariance principle for an observable ψ on $(\mathbb{P}^k, f, \mu)$ by applying Philipp-Stout's theorem for $\chi = \psi \circ \omega$ on (Σ, s, ν). The invariance principle implies the Central Limit Theorem as well as several statistical properties for the class $\mathcal{U}$. As an application, we give a *direct* proof of the absolute continuity of the measure μ when it satisfies Pesin's formula. This approach relies on the Central Limit Theorem for the unbounded observable $\log \mathsf{Jac}\, f \in \mathcal{U}$.

Key Words: holomorphic dynamics, Bernoulli coding map, almost sure invariance principle.
MSC: 37F10, 37C40, 60F17.

1 Introduction

Let $f : \mathbb{P}^k \to \mathbb{P}^k$ be an holomorphic endomorphism of algebraic degree $d \geq 2$. Its equilibrium measure μ is the limit of the probability measures $d_t^{-n}(f^n)^*\eta^k$, where $d_t := d^k$ is the topological degree of f and η^k is the standard volume form on $\mathbb{P}^k$. We refer to the survey article of Sibony [Sib] for an introduction to the dynamical systems $(\mathbb{P}^k, f, \mu)$. Fornaess-Sibony proved that μ is mixing [FS] and Briend-Duval that μ is the unique measure of maximal entropy [BrDu2].

Przytycki-Urbański-Zdunik [PUZ] introduced coding techniques for $(\mathbb{P}^1, f, \mu)$. This allowed them to prove the Almost Sure Invariance Principle (ASIP) for Hölder and singular observables, like $\log |f'|$. In the present article, we extend the coding techniques to $(\mathbb{P}^k, f, \mu)$ and obtain the ASIP for observables which allow analytic singularities. As an application, we obtain a direct proof of the absolute continuity of μ when it satisfies Pesin's formula. We review our results in subsections 1.1 - 1.4, subsection 1.5 is devoted to related results.

1.1 Bernoulli coding maps

Let us endow $\Sigma := \{1, \ldots, d_t\}^{\mathbb{N}}$ with the natural product measure $\nu := \otimes_{n=0}^{\infty} \bar{\nu}$, where $\bar{\nu}$ is equidistributed on $\{1, \ldots, d_t\}$. We denote by $\tilde{\alpha}$ the elements of Σ and by s the left shift acting on Σ. Let $\mathcal{J}$ be the support of μ. The following theorem yields coding maps $\omega : \Sigma \to \mathcal{J}$ up to zero measure sets. The set $\mathcal{S} \subset \mathbb{P}^k$ will be defined in section 4, it has zero Lebesgue measure.

Theorem A: *Let $z \in \mathbb{P}^k \setminus \mathcal{S}$. There exist an s-invariant set $\Sigma' \subset \Sigma$ of full ν-measure and an f-invariant set $\mathcal{J}' \subset \mathcal{J}$ of full μ-measure satisfying the following properties. For any $\tilde{\alpha} \in \Sigma'$, the point $\omega(\tilde{\alpha}) := \lim_{n\to\infty} z_n(\tilde{\alpha}) \in \mathcal{J}'$ is well defined. We have $\omega_* \nu = \mu$ and the following diagram commutes:*

$$\begin{tikzcd} \Sigma' \arrow[r, "s"] \arrow[d, "\omega"'] & \Sigma' \arrow[d, "\omega"] \\ \mathcal{J}' \arrow[r, "f"] & \mathcal{J}' \end{tikzcd}$$

Moreover there exist $\theta, \epsilon > 0$, $n_z \geq 1$ and $\tilde{n} : \Sigma' \to \mathbb{N}$ larger than n_z such that:

1. *$d(z_n(\tilde{\alpha}), \omega(\tilde{\alpha})) \leq \tilde{c}_\epsilon \, d^{-\epsilon n}$ for every $\tilde{\alpha} \in \Sigma'$ and $n \geq \tilde{n}(\tilde{\alpha})$,*
2. *$\nu(\{\tilde{n} \leq q\}) \geq 1 - c_\theta \, d^{-\theta q}$ for every $q \geq n_z$.*

We note that Σ', $\mathcal{J}'$ and ω depend on $z \in \mathbb{P}^k \setminus \mathcal{S}$, but $\omega_* \nu = \mu$ holds true for any such z. Observe also that ω is not necessarily injective. The proof of theorem A (see section 4) is based on the construction of a geometric coding tree, following the approach of Przytycki-Urbański-Zdunik [PUZ] for $(\mathbb{P}^1, f, \mu)$. The point z is the *root* of the tree, and the set $\{z_n(\tilde{\alpha}), \tilde{\alpha} \in \Sigma\}$ is a suitable enumeration of the d_t^{n+1} points of $f^{-(n+1)}(z)$, these are *vertices* of the tree. The convergence of $(z_n(\tilde{\alpha}))_n$ for a generic $\tilde{\alpha} \in \Sigma$ is obtained by constructing d_t good paths joining z to $w \in f^{-1}(z)$, whose inverse images decrease exponentially. In the context of $(\mathbb{P}^1, f, \mu)$, that property was obtained in [PUZ] by using Koebe distortion theorem. The difficulty in higher dimensions is to substitute this argument. We establish for that purpose a quantified version of a theorem of Briend-Duval (see section 3).

1.2 The class $\mathcal{U}$ and approximation by cylinders

Definition : *An observable $\psi : \mathbb{P}^k \to \mathbb{R} \cup \{-\infty\}$ belongs to the class $\mathcal{U}$ if:*
- *e^ψ is h-Hölder for some $h > 0$,*
- *$\mathcal{N}_\psi := \{\psi = -\infty\}$ is a (possibly empty) proper algebraic set of $\mathbb{P}^k$,*
- *$\psi \geq \log d(\cdot, \mathcal{N}_\psi)^\rho$ for some $\rho > 0$.*

For instance, the Hölder functions are in $\mathcal{U}$, as well as the unbounded function $\log \mathsf{Jac}\, f$. We will show that $\mathcal{U} \subset L^p(\mu)$ for any $1 \leq p < +\infty$ (see subsection 2.2).

Theorem B: *Let $\psi \in \mathcal{U}$ be a μ-centered observable and ω be a coding map provided by theorem A. Let $\chi := \psi \circ \omega$ and $1 \leq p < +\infty$. We denote by $\mathbb{E}(\chi|\mathcal{C}_n)$ the conditional expectation of χ with respect to the $(n+1)$-cylinders.*

1. *there exist $\hat{c}_p, \lambda_p > 0$ such that $\| \chi - \mathbb{E}(\chi|\mathcal{C}_n) \|_p \leq \hat{c}_p \, e^{-n\lambda_p}$ for every $n \geq 0$.*

2. *$R_j(\chi) := \int_\Sigma \chi \cdot \chi \circ s^j \, d\nu$ satisfies $|R_j(\chi)| \leq 2 \| \chi \|_2 \, \hat{c}_2 \, e^{-(j-1)\lambda_2}$ for every $j \geq 1$.*

The proof occupies section 5, it is based on the regularity properties of ω (namely the points 1, 2 of theorem A) and on the fact that μ is a Monge-Ampère mass with Hölder potentials. Theorem B allows us to prove theorem C below.

1.3 Almost Sure Invariance Principle

Let $\psi \in L^2(\mu)$ be a μ-centered observable and $S_n(\psi) := \sum_{j=0}^{n-1} \psi \circ f^j$. We say that ψ satisfies the Almost Sure Invariance Principle (ASIP) if there exist, on an extended probability space, a sequence of random variables $(\mathcal{S}_n)_{n \geq 0}$ together with a Brownian motion $\mathcal{W}$ such that for some $\gamma > 0$:

- $\mathcal{S}_n = \mathcal{W}(n) + o(n^{1/2-\gamma})$ almost everywhere,

- $(S_0(\psi), \ldots, S_n(\psi))$ and $(\mathcal{S}_0, \ldots, \mathcal{S}_n)$ have the same distribution for any $n \geq 0$.

We shall denote σ-ASIP to specify the variance of Brownian motion.

Theorem C: *For every μ-centered observable $\psi \in \mathcal{U}$, we have:*

1. *$\sigma := \lim_{n \to \infty} \frac{1}{\sqrt{n}} \| S_n(\psi) \|_2$ exists, and $\sigma^2 = \int_{\mathbb{P}^k} \psi^2 \, d\mu + 2 \sum_{j \geq 1} \int_{\mathbb{P}^k} \psi \cdot \psi \circ f^j \, d\mu$.*

2. *If $\sigma = 0$, then $\psi = u - u \circ f$ holds μ-a.e. for some $u \in L^2(\mu)$.*

3. *If $\sigma > 0$, then ψ satisfies the σ-ASIP.*

The ASIP implies classical limit theorems related to Brownian motion: the Central Limit Theorem (CLT), the Law of Iterated Logarithm, Kolmogorov integral tests (see [De], [PS]). The ASIP also implies the almost sure version of the CLT, meaning that $\frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \delta_{\frac{1}{\sqrt{k}} S_k(\psi)(x)}$ converges μ-a.e. to the normal law $\mathcal{N}(0, \sigma^2)$ (see [LP], [CG]).

Let us outline the proof of theorem C (see section 6). Let $\omega : \Sigma \to \mathbb{P}^k$ be a coding map provided by theorem A and $\psi \in \mathcal{U}$. Since ω satisfies $f \circ \omega = \omega \circ s$ and $\omega_* \nu = \mu$, we are reduced to prove the assertions for $\chi = \psi \circ \omega$ on (Σ, s, ν). The points 1 and 2 follow from theorem B(2) and classical arguments. The point 3 is a consequence of theorem B(1) and Philipp-Stout's theorem ([PS], section 7). That result relies on an approximation of the partial sums of $(\chi \circ s^j)_{j \geq 0}$ by a sequence of martingale differences defined with respect to the increasing filtration $(\mathcal{C}_n)_{n \geq 0}$.

1.4 An application to smooth ergodic theory

Let $\chi_1 \le \ldots \le \chi_k$ be the Lyapunov exponents of μ. Briend-Duval [BrDu1] proved that they are larger than or equal to $\log d^{1/2}$. Since μ has entropy $\log d^k$, Pesin's formula $h(\mu) = 2(\chi_1 + \ldots + \chi_k)$ holds if and only if these exponents are minimal. We proved in a previous article that μ is then absolutely continuous with respect to Lebesgue measure [Du]. We there followed the classical approach of Sinai-Pesin-Ledrappier, based on the construction of a suitable invariant partition which is dilated and realizes entropy (see [P1], [Le]). We propose in section 7 a new proof, based on the CLT for the unbounded μ-centered observable $J := \log \mathsf{Jac}\, f - 2(\chi_1 + \ldots + \chi_k) \in \mathcal{U}$. We obtain the following result, where $\sigma_J := \lim_{n\to\infty} \frac{1}{\sqrt{n}} \parallel S_n(J) \parallel_2$.

Theorem D: *If the Lyapunov exponents are minimal equal to* $\log d^{1/2}$*, then* $\sigma_J = 0$*, and* μ *is absolutely continuous with respect to Lebesgue measure.*

A crucial fact for the proof is that for any holomorphic endomorphism of $\mathbb{P}^k$ and any μ-generic point $x \in \mathbb{P}^k$, the minimal dilation rate of f^n at x (i.e. $\parallel (d_x f^n)^{-1} \parallel^{-1}$) is bounded below by $d^{n/2}$ up to the multiplicative factor $1/n$. In other words, the usual $e^{-n\epsilon}$-correction, due to the non-uniform hyperbolicity of $(\mathbb{P}^k, f, \mu)$, can be replaced here by $1/n$. This was proved by Berteloot-Dupont [BeDu], using a pluripotential result of Briend-Duval [BrDu1] and the fact that μ is a Monge-Ampère mass. In particular, the product of the dilation rates satisfies $\mathsf{Jac}\, f^n(x) \ge \parallel (d_x f^n)^{-1} \parallel^{-2k} \ge (d^{n/2}/n)^{2k} = d^{kn}/n^{2k}$. Now if we assume $\sigma_J > 0$, then the function $\log \mathsf{Jac}\, f^n$ would present non trivial oscillations around its mean value $\log d^{kn}$, due to the CLT. More precisely, it would imply $\log \mathsf{Jac}\, f^n \le \log d^{kn} - \sigma_J \sqrt{n}$ on a subset of μ-measure $\simeq \int_{-\infty}^{-1} e^{-u^2/2}$. That contradicts the preceding estimate, hence $\sigma_J = 0$. We deduce the absolute continuity of μ from the cocycle property $J = u - u \circ f$ μ-a.e. and a linearization property of the dynamics along typical negative orbits [BeDu].

1.5 Related results

The systems $(\mathbb{P}^k, f, \mu)$ and (Σ, s, ν) are actually conjugated by a bimeasurable map up to zero measure subsets, that property was proved by Briend [Br]. However, the regularity of the conjugacy seems difficult to handle. Let us also mention that finite-to-one coding maps $(\mathbb{P}^k, f, \mu) \to (\Sigma, s, \nu)$ were constructed by Buzzi [Bu] by means of suitable partitions of $\mathbb{P}^k$.

The ASIP has been proved for many dynamical systems: for piecewise monotonic maps by Hofbauer-Keller [HK], for Anosov maps by Denker-Philipp [DP] and for partially and non-uniformly hyperbolic systems by Dolgopyat [Do] and Melbourne-Nicol [MN]. We refer to the survey articles of Chernov [C] and Denker [De] for limit theorems and statistical properties concerning dynamical systems.

The ASIP implies the CLT. Nevertheless, the latter can be directly proved via coding techniques and Ibragimov's theorem [I]. That method was employed by Sinai

[Sin] and Ratner [R] for the geodesic flow in negative curvature, and by Bowen [Bo] for Anosov maps. In the present article, Ibragimov's condition is fulfilled by theorem B.

The Gordin's theorem provides another method for proving the CLT (see [G], [Li]). It relies on an approximation of $(\psi \circ f^j)_{j\geq 0}$ by a sequence of reverse martingale differences. In our context, this can be done if $\sum_{n\geq 0} \| \Lambda^n \psi \|_2$ (denoted $(\star)$) converges, where Λ denotes the Ruelle-Perron-Frobenius operator (we have $\Lambda^n\psi(z) = \frac{1}{d_t^n}\sum_{y\in f^{-n}(z)} \psi(y)$ for every $z \in \mathbb{P}^k$). Let us note that the reverse martingale mentioned is defined with respect to the decreasing filtration $(f^{-n}\mathcal{B})_{n\geq 0}$, where $\mathcal{B}$ is the Borel σ-algebra of $\mathbb{P}^k$.

The exponential decay of correlations ensures the convergence of $(\star)$. This was proved in the context of $(\mathbb{P}^k, f, \mu)$ by Fornaess-Sibony [FS] for C^2 observables and by Dinh-Sibony for Hölder observables [DS2]. Dinh-Nguyen-Sibony have recently extended that property for differences of quasi-plurisubharmonic functions (the so-called *dsh* functions) [DNS2]. The proof relies on exponential estimates for plurisubharmonic functions with respect to μ. They also obtained in that article a Large Deviations Theorem for bounded dsh and Hölder observables. In [DNS1], Dinh-Nguyen-Sibony proved the local CLT for $(\mathbb{P}^1, f, \mu)$ by using the theory of perturbed operators.

Denker-Przytycki-Urbański [DPU] employed a geometric method to prove the convergence of $(\star)$ for $(\mathbb{P}^1, f, \mu)$ and Hölder observables. The idea was to compare $\Lambda^n\psi(z)$ to $\Lambda^n\psi(z')$ by using the contraction of most of the inverse branches of f^n. The cornerstone is a precise analysis of the dynamics near the critical points in the support of μ. Cantat-Leborgne [CL] extended this approach to $(\mathbb{P}^k, f, \mu)$. A crucial ingredient was a polynomial estimate for the μ-measure of postcritical neighbourhoods (lemma 5.7 of [CL]). The original proof of that lemma contains a gap, the authors have recently proposed another one. Cantat-Leborgne also established in [CL] a quantified version of the Briend-Duval theorem. Our version is similar, but we shall give a different proof.

The systems $(\mathbb{P}^k, f, \mu)$ whose measure μ is absolutely continuous with respect to Lebesgue measure were characterized by Berteloot, Dupont and Loeb [BeDu], [BL]. In that case, f is semi-conjugated to an affine dilation on a complex torus, these maps are the so-called *Lattès examples*. We note that theorem D characterizes these maps by the minimality of the Lyapunov exponents. Another characterization of Lattès examples involves the *Hausdorff dimension* of μ, defined as the infimum of the Hausdorff dimension of Borel sets with full μ-measure (see Pesin's book [P2]): Dinh-Dupont [DD] proved that $\dim_{\mathcal{H}}(\mu) = 2k$ if and only if the exponents are minimal. In the context of $(\mathbb{P}^1, f, \mu)$, Mañé [Mañ] proved that $\log d = \dim_{\mathcal{H}}(\mu)\cdot\chi$, where χ denotes the Lyapunov exponent of μ. In particular, the function $L := \log d - \dim_{\mathcal{H}}(\mu)\cdot \log|f'|$ is a μ-centered observable. Zdunik [Z] proved that $\sigma_L = 0$ if and only if f is a Lattès example, a Tchebychev polynomial or a power $z^{\pm d}$. The proof relies on the classification of critically finite fractions with parabolic Thurston's orbifold.

2 Generalities

2.1 The holomorphic systems $(\mathbb{P}^k, f, \mu)$

We introduce in this section the systems $(\mathbb{P}^k, f, \mu)$. We refer to the articles [BrDu1], [BrDu2], [FS] and [Sib] for definitions and properties. Here $\mathbb{P}^k$ denotes the complex projective space of dimension k. We denote by η the Fubini-Study form on $\mathbb{P}^k$. This is a $(1,1)$-form defined in homogeneous coordinates by $\frac{i}{2\pi}\partial\bar{\partial}\log \| z \|^2$. It induces the standard metric on $\mathbb{P}^k$, the volume of $\mathbb{P}^k$ with respect to this metric is equal to 1. The form η induces on every complex line $L \subset \mathbb{P}^k$ the spherical metric with area 1. Let f be an holomorphic endomorphism of $\mathbb{P}^k$ with algebraic degree $d \geq 2$. It is defined in homogeneous coordinates by $[P_0 : \cdots : P_k]$ where the P_i are homogeneous polynomials of degree d (without common zero except the origin). The topological degree of f is $d_t := d^k$. An inverse branch of f^n on $U \subset \mathbb{P}^k$ is an injective holomorphic map g_n satisfying $f^n \circ g_n = \mathrm{Id}_U$. We let $\mathsf{Per}f := \cup_{n\geq 1}\{x \in \mathbb{P}^k \,,\, f^n(x) = x\}$, this set is at most countable. Let $\mathcal{C}$ be the critical set of f, $\mathcal{V} := \cup_{i=0}^{\infty} f^i(\mathcal{C})$ and $\mathcal{V}_n := \cup_{i=1}^{n} f^i(\mathcal{C})$. The degree of $\mathcal{V}_n$, denoted τ_n, is equal to $(d + \ldots + d^n)\deg \mathcal{C}$ counted with multiplicity.

The equilibrium measure μ is defined as the limit of $\mu_{n,z} := \frac{1}{d_t^n}\sum_{f^n(y)=z}\delta_y$, where δ_y denotes the Dirac mass at y. In that definition, z has to be taken outside a totally invariant algebraic set $\mathcal{E} \subset \mathcal{V}$, the so-called exceptional set of f. We denote by $\mathcal{J}$ the support of μ. The measure μ is mixing and satisfies $\mu(f(B)) = d_t\, \mu(B)$ whenever f is injective on B. It is the unique measure of maximal entropy (equal to $\log d_t$). The Lyapunov exponents $\chi_1 \leq \ldots \leq \chi_k$ of μ are larger than or equal to $\log d^{1/2}$. They satisfy the classical formula $\int_{\mathbb{P}^k} \log \mathsf{Jac}\, f \, d\mu = 2(\chi_1 + \ldots + \chi_k)$, where $\mathsf{Jac}\, f$ is the non-negative $\mathcal{C}^\infty$ function on $\mathbb{P}^k$ satisfying $f^*\eta^k = \mathsf{Jac}\, f \cdot \eta^k$. The latter is the real jacobian of f, it vanishes on the critical set $\mathcal{C}$ of f.

The measure μ can also be defined *via* pluripotential theory: we have $\mu = T^k$, where T is the Green current of f. The latter is a closed positive $(1,1)$ current on $\mathbb{P}^k$ with Hölder potentials. In particular, for any algebraic subset $A \subset \mathbb{P}^k$, there exist $c, \gamma > 0$ such that the r-neighbourhood of A satisfies $\mu(A[r]) \leq c\, r^\gamma$ for any $r > 0$ (see [DS4], Prop. 2.3.7). For any $\delta > 0$ and $\tilde{c} > 0$, we set $c_\delta := (1 - d^{-\delta})^{-1}$ and $\tilde{c}_\delta := \tilde{c}(1 - d^{-\delta})^{-1}$. In the sequel, $c > 0$ is a constant independent of n, it may differ from a line to another.

2.2 The class $\mathcal{U}$

Let us recall the definition of the class $\mathcal{U}$ (see subsection 1.2).

Definition 2.1 *Let $\mathcal{U}$ be the set of functions $\psi : \mathbb{P}^k \to \mathbb{R} \cup \{-\infty\}$ satisfying:*
- *e^ψ is h-Hölder on $\mathbb{P}^k$ for some $h > 0$,*
- *$\mathcal{N}_\psi := \{\psi = -\infty\}$ is a (possibly empty) proper algebraic set of $\mathbb{P}^k$,*
- *$\psi \geq \log d(\cdot, \mathcal{N}_\psi)^\rho$ on $\mathbb{P}^k$ for some $\rho > 0$.*

The Hölder functions belong to $\mathcal{U}$. Examples of unbounded observables are:

- the functions $\psi = \log|Q| - q\log\|\cdot\|$, where Q is a q-homogeneous polynomial on $\mathbb{C}^{k+1}$. Here the algebraic subset $\mathcal{N}_\psi$ is the zero set of Q.

- the functions $\psi = \log \| \Lambda^j d_x f \|$ $(1 \le j \le k)$, where $\Lambda^j d_x f$ is the j-exterior power of the differential $d_x f$. In particular, $\log \mathsf{Jac}\, f \in \mathcal{U}$ (take $j = k$).

The conditions of definition 2.1 are easy to verify for these functions, the last one is a consequence of Lojasiewicz's inequality (see [Lo], §4.7). We prove below that $\psi \in L^p(\mu)$ for any $\psi \in \mathcal{U}$ and $1 \le p < +\infty$. Actually, we establish an estimate for $\int_{\mathcal{N}_\psi[r]} |\psi|^p$, useful to prove theorem B. We recall that $\mu(\mathcal{N}_\psi[r]) \le c\, r^\gamma$ for some $c, \gamma > 0$ (see subsection 2.1).

Proposition 2.2 *Let $\psi \in \mathcal{U}$ and $1 \le p < +\infty$. There exists $\kappa > 0$ such that:*

$$\forall\, 0 < r < 1/2 \quad , \quad \int_{\mathcal{N}_\psi[r]} |\psi|^p \, d\mu \le \kappa\, r^{\gamma/2}.$$

In particular $\psi \in L^p(\mu)$.

Proof: Let $\psi \in \mathcal{U}$ and $\mathcal{N} := \mathcal{N}_\psi$. We may assume that $0 \le e^\psi \le 1$ by adding some constant to ψ. Let $r < 1/2$ and $\mathcal{Q}_j := \mathcal{N}[r/2^j] \setminus \mathcal{N}[r/2^{j+1}]$. Since $e^\psi \ge (r/2^{j+1})^\rho$ on $\mathcal{Q}_j$, we obtain:

$$\int_{\mathcal{N}[r]} |\psi|^p \, d\mu = \sum_{j\ge 0} \int_{\mathcal{Q}_j} |\log e^\psi|^p \, d\mu \le \sum_{j \ge 0} \left| \rho \log\left(\frac{r}{2^{j+1}}\right) \right|^p \cdot \mu(\mathcal{Q}_j).$$

The inequalities $\mu(\mathcal{Q}_j) \le c(r/2^j)^\gamma$ and $|\log \frac{r}{2^{j+1}}| = (j+1)\log 2 + \log\frac{1}{r} \le (j+2)\log\frac{1}{r}$ yield:

$$\int_{\mathcal{N}[r]} |\psi|^p \, d\mu \le \left[c\,\rho^p \sum_{j \ge 0} \frac{(j+2)^p}{2^{\gamma j}} \right] \left(\log\frac{1}{r} \right)^p r^\gamma = M_{\rho,\gamma} \cdot \left(\log \frac{1}{r}\right)^p r^{\gamma/2} \cdot r^{\gamma/2}.$$

The lemma follows with $\kappa := M_{\rho,\gamma} \cdot \sup_{0<r<1/2} \left(\log\frac{1}{r}\right)^p r^{\gamma/2}$. □

2.3 The Bernoulli space (Σ, s, ν)

We endow $\mathcal{A} := \{1, \ldots, d_t\}$ with the equidistributed probability measure $\bar{\nu}$. We set $\Sigma := \mathcal{A}^{\mathbb{N}}$, $s : \Sigma \to \Sigma$ the left shift and $\nu := \otimes_{n=0}^{\infty} \bar{\nu}$. We denote by $\tilde{\alpha} := (\alpha_n)_{n \ge 0}$ the elements of Σ, by $\mathcal{C}_n$ the set of cylinders of length $n+1$, and by $\pi_n : \Sigma \to \mathcal{A}^{n+1}$ the projection $\pi_n(\tilde{\alpha}) := (\alpha_0, \ldots, \alpha_n)$. For any $\tilde{\alpha} \in \Sigma$, we set $C_n(\tilde{\alpha}) := \pi_n^{-1}(\alpha_0, \ldots, \alpha_n)$. We denote by $\mathbb{E}(\chi|\mathcal{C}_n)$ the conditional expectation of $\chi \in L^2(\nu)$ with respect to $\mathcal{C}_n$. If $\mathcal{L} = \{A_1, \ldots, A_p\} \subset \mathcal{C}_n$, we set $\mathcal{L}^* := \cup_{1 \le j \le p} A_j$.

2.4 Almost Sure Invariance Principle

Let (X, g, m) be either (Σ, s, ν) or $(\mathbb{P}^k, f, \mu)$. For any observable $\varphi \in L^2(m)$, we set $S_n(\varphi) := \sum_{j=0}^{n-1} \varphi \circ g^j$ and $R_j(\varphi) := \int_X \varphi \cdot \varphi \circ g^j \, dm$. We say that φ is m-centered if $\int_X \varphi \, dm = 0$ and that φ is a cocycle if $\varphi = u - u \circ g$ m-a.e. for some $u \in L^2(m)$.

An observable φ on (X, g, m) satisfies the Almost Sure Invariance Principle (ASIP) if there exist on a probability space $(\tilde{X}, \tilde{m})$ a sequence of random variables $(\mathcal{S}_n)_{n \geq 0}$ and a Brownian motion $\mathcal{W}$ such that:

- $\mathcal{S}_n = \mathcal{W}(n) + o(n^{1/2-\gamma})$ $\tilde{m}$-a.e. for some $\gamma > 0$,
- $(S_0(\psi), \ldots, S_n(\psi))$ and $(\mathcal{S}_0, \ldots, \mathcal{S}_n)$ have the same distribution for any $n \geq 0$.

We denote σ-ASIP to specify the variance of Brownian motion. The σ-ASIP implies the σ-Central Limit Theorem (σ-CLT), meaning that:

$$\forall t \in \mathbb{R} \ , \ \lim_{n \to \infty} m \left(\frac{S_n(\varphi)}{\sigma \sqrt{n}} \leq t \right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-u^2/2} \, du.$$

Remark 2.3 *Suppose that $\omega : \Sigma \to \mathbb{P}^k$ is a coding map provided by theorem A. Since $\omega_* \nu = \mu$ and $f \circ \omega = \omega \circ s$, a μ-centered observable $\psi \in L^2(\mu)$ satisfies the σ-ASIP if and only if the ν-centered observable $\chi := \psi \circ \omega \in L^2(\nu)$ satisfies the σ-ASIP.*

We shall use Philipp-Stout's theorem ([PS], Section 7) to prove the ASIP for $\chi := \psi \circ \omega$ on the Bernoulli space (Σ, s, ν). The version below comes from the original one by using the s-invariance of ν and the independence of the random process $(\xi_n)_{n \geq 0}$ defined by $\xi_n(\tilde{\alpha}) = \alpha_n$.

Theorem (Philipp-Stout) *Let χ be a ν-centered observable on Σ satisfying:*

1. *$\chi \in L^{2+\delta}(\nu)$ for some $\delta > 0$,*
2. *$\| \chi - \mathbb{E}(\chi | \mathcal{C}_n) \|_{2+\delta} \leq c \, \beta^n$ for some $c > 0$ and $\beta < 1$.*

Then the sequence $\frac{1}{\sqrt{n}} \| S_n(\chi) \|_2$ has a limit σ. If $\sigma > 0$, then χ satisfies the σ-ASIP.

Let us compare that result with Ibragimov's theorem (see [I], Theorem 2.1), which only requires moments of order 2 and a summability condition:

Theorem (Ibragimov) *Let χ be a ν-centered observable on Σ satisfying:*

$$\sum_{n \geq 0} \| \chi - \mathbb{E}(\chi | \mathcal{C}_n) \|_2 < \infty.$$

Then the sequence $\frac{1}{\sqrt{n}} \| S_n(\chi) \|_2$ has a limit σ. If $\sigma > 0$, then χ satisfies the σ-CLT.

3 A quantified version of Briend-Duval theorem

This section is devoted to the proof of theorem 3.2 (see subsection 3.2). That result will be crucial to establish theorem A.

3.1 Briend-Duval theorem

We recall that $\mathcal{V}_l = \cup_{i=1}^{l} f^i(\mathcal{C})$, $\mathcal{V} = \cup_{i=0}^{\infty} f^i(\mathcal{C})$ and that $d_t = d^k$ is the topological degree of f (see subsection 2.1). We set $\tau_* := 2 \deg \mathcal{V}_1/(1-1/d)$.

Theorem (Briend-Duval [BrDu2]) *Let $\eta > 0$ and $l \geq 1$ be such that $\tau_*/d^l < \eta$. Let L be a complex line in $\mathbb{P}^k$ not contained in $\mathcal{V}$, and $\Delta \Subset \tilde{\Delta}$ be topological discs in $L \setminus \mathcal{V}_l$. Then, for any $n \geq l$, there exist $(1-\eta)d_t^n$ inverse branches g_n on Δ satisfying:*

$$\mathit{diam}\, g_n(\Delta) \leq \frac{\tilde{c}\, d^{-n/2}}{\eta^{1/2}\ \mathit{mod}\,(\tilde{\Delta} \setminus \Delta)^{1/2}},$$

where $\tilde{c}$ is a universal constant, and $\mathit{mod}\,(\tilde{\Delta} \setminus \Delta)$ is the modulus of the annulus $\tilde{\Delta} \setminus \Delta$.

Let us recall the definition of the modulus (see Ahlfors book [A], chapters 1 and 2). Let Λ denote the family of curves joining the boundary components of $A := \tilde{\Delta} \setminus \Delta$. For any conformal metric ρ on A, we respectively denote by area_ρ and by l_ρ the area and the length with respect to ρ. We denote by $\mathsf{conf}\,(A)$ the set of conformal metrics giving finite area to A. The modulus of the annulus A is then defined by:

$$\mathsf{mod}\,(A) := \sup_{\rho \in \mathsf{conf}\,(A)} \frac{l_\rho(\Lambda)^2}{\mathsf{area}_\rho(A)},$$

where $l_\rho(\Lambda) := \inf_{\lambda \in \Lambda} l_\rho(\lambda)$.

3.2 Statement of the quantified version

We begin with some notations. Let $0 < \theta < 1$ and $\theta_n := [\theta n + \frac{\log \tau_*}{\log d}] + 1$. We introduce this integer in view of applying Briend-Duval theorem with $\eta = d^{-\theta n}$ and $l = \theta_n$ (indeed, $\tau_*/d^{\theta_n} < d^{-\theta n}$). Since the degree of $\mathcal{V}_{\theta_n} = \cup_{i=1}^{\theta_n} f^i(\mathcal{C})$ is at most $\tau_{\theta_n} = (d + \ldots + d^{\theta_n}) \deg \mathcal{C}$, we have $\tau_{\theta_n} < d^{\theta_n}$ up to a multiplicative constant.

We let $0 < \theta < \theta' < 1$ and consider $n_0 \geq 1$ satisfying:

$$\forall n \geq n_0 \quad , \quad \theta_n < \theta' n \quad \text{and} \quad \tau_{\theta_n} < d^{\theta' n}. \tag{1}$$

Let us recall that $\mathcal{V}_{\theta_n}[\delta]$ is the δ-neighbourhood of $\mathcal{V}_{\theta_n}$ in $\mathbb{P}^k$. We fix $\theta'/2 < \zeta < 1$ and define $\mathcal{D} := \limsup_{n \geq n_0} \mathcal{V}_{\theta_n}[d^{-\zeta n}]$.

Proposition 3.1 *The set $\mathcal{D}$ satisfies* $\mathrm{Vol}\,(\mathcal{D}) = 0$.

The proof is postponed to subsection 3.5. We now state the quantified version. The constant $\tilde{c}$ has been introduced in the statement of Briend-Duval's theorem, and we denote by L the complex line containing z and w.

Theorem 3.2 *There exists $\epsilon > 0$ such that for every distinct points $(z, w) \notin \mathcal{D} \cup \mathcal{V}$, there exist an injective smooth path $\gamma : [0, 1] \to L \setminus \mathcal{V}$ joining z and w, a decreasing family of topological discs $(\Delta_n)_n \subset L$ and an integer $n_{z,w}$ such that for any $n \geq n_{z,w}$:*

1. *$\gamma[0, 1] \subset \Delta_n \subset L \setminus \mathcal{V}_{\theta_n}$,*
2. *there exist $(1 - d^{-\theta n})d_t^n$ inverse branches of f^n on Δ_n,*
3. *these branches satisfy $\mathsf{diam}\, g_n(\Delta_n) \leq \tilde{c}\, d^{-\epsilon n}$.*

We note that θ, ϵ and $\tilde{c}$ do not depend on $(z, w) \in \mathbb{P}^k \setminus (\mathcal{D} \cup \mathcal{V})$.

3.3 Construction of good paths in the complex line $L \subset \mathbb{P}^k$

Let (z, w) be distinct points in $\mathbb{P}^k \setminus (\mathcal{D} \cup \mathcal{V})$. We identify the complex line L containing z and w with the 2-dimensional sphere. We recall that the Fubini-Study metric induces on L the standard spherical metric s with area 1. We assume with no loss of generality that z and w are the North and South pole of L. Let E be the equator of L. For any $y \in E$, we denote by M_y the meridian containing y, and by $M_y\{\delta\}$ the δ-neighbourhood of M_y in L for the spherical metric. The constants $0 < \theta < \theta' < 2\zeta$ have been defined in subsection 3.2. Now we let $0 < \zeta < \zeta' < \zeta'' < 1$ satisfying:

$$\theta' < \zeta'' - \zeta' \text{ and } \theta + \zeta'' < 1. \tag{2}$$

We may take for $(\theta, \theta', \zeta, \zeta', \zeta'')$ suitable multiples of a small $\theta > 0$. The second inequality of (2) will be used in next subsection. The integer n_0 has been defined in subsection 3.2.

Proposition 3.3 *Let (z, w) be distinct points in $\mathbb{P}^k \setminus (\mathcal{D} \cup \mathcal{V})$. With the above notations, there exists a subset $F \subset E$ of full Lebesgue measure satisfying the following properties. For any $y \in F$, there exists $n_{z,w}(y) \geq n_0$ such that:*

1. *the meridian M_y does not intersect $\mathcal{V}$,*
2. *the neighbourhood $M_y\{d^{-\zeta'' n}\}$ does not intersect $\mathcal{V}_{\theta_n}$ for any $n \geq n_{z,w}(y)$.*

Let us now prove proposition 3.3. We start with some notations. Let H^+ and H^- be the (open) North and South hemispheres of L, these sets induce a partition $L = H^+ \sqcup E \sqcup H^-$. We denote by Leb the Lebesgue measure on E and by p_1 (resp. p_2) the spherical projection from z (resp. w) to E. For any $y \in E$ and $\delta > 0$, let $\mathcal{I}(y, \delta)$ be the interval in E centered at y with length 2δ. We also denote by $D(c, \delta) \subset L$ the disc with center c and radius δ. We define $p_\kappa(c) := p_1(c)$ if $c \in H^+ \cup E$ and $p_\kappa(c) := p_2(c)$ if $c \in H^-$. The same convention holds for the projection of $D(c, \delta)$ to E: we use p_1 or

p_2 depending on $c \in H^+ \cup E$ or $c \in H^-$.

Let $\{c_i\,,\, 1 \leq i \leq l_{\theta_n}\} := \mathcal{V}_{\theta_n} \cap L$, where $l_{\theta_n} \leq \deg(\mathcal{V}_{\theta_n}) \leq \tau_{\theta_n}$. Since the Fubini-Study metric induces s on L, the set $\mathcal{L}_{\theta_n} := \cup_{i=1}^{l_{\theta_n}} D(c_i, d^{-\zeta n})$ is a subset of $\mathcal{V}_{\theta_n}[d^{-\zeta n}]$. We recall that $\mathcal{D} = \limsup_{n \geq n_0} \mathcal{V}_{\theta_n}[d^{-\zeta n}]$ and that $(z,w) \notin \mathcal{D}$. Thus there exists $n_1 \geq n_0$ depending on (z,w) such that:

$$\forall n \geq n_1\ ,\ (z,w) \notin \mathcal{V}_{\theta_n}[d^{-\zeta n}]. \tag{3}$$

In particular $(z,w) \notin \mathcal{L}_{\theta_n}$. Since $\zeta < \zeta' < \zeta''$, we may increase n_1 so that $d^{-\zeta' n} + d^{-\zeta'' n} < d^{-\zeta n}$ for any $n \geq n_1$. We have therefore, for $\rho = z$ or w:

$$\forall\, 1 \leq i \leq l_{\theta_n}\ ,\ \forall n \geq n_1\ ,\ D(\rho, d^{-\zeta' n}) \cap D(c_i, d^{-\zeta'' n}) = \emptyset.$$

This implies, with $e_i := p_\kappa(c_i) \in E$ and c a positive constant:

$$\forall\, 1 \leq i \leq l_{\theta_n}\ ,\ p_\kappa\left(D(c_i, d^{-\zeta'' n})\right) \subset \mathcal{I}_i := \mathcal{I}(e_i\,,\, c\, d^{-\zeta'' n} \cdot d^{\zeta' n}). \tag{4}$$

Hence $\mathcal{I}(\theta_n) := \cup_{i=1}^{l_{\theta_n}} \mathcal{I}_i$ satisfies $\mathsf{Leb}\,\mathcal{I}(\theta_n) \leq \tau_{\theta_n} \cdot c\, d^{-(\zeta''-\zeta')n} \leq c\, d^{(\theta' - (\zeta''-\zeta'))n}$. Since $\sum_n \mathsf{Leb}\,\mathcal{I}(\theta_n) < \infty$ (see (2)), the Borel-Cantelli lemma yields, for every y in a full Lebesgue measure subset $F' \subset E$, an integer $n_{z,w}(y) \geq n_1$ satisfying:

$$y \notin \bigcup_{n \geq n_{z,w}(y)} \mathcal{I}(\theta_n). \tag{5}$$

Let us prove the point 2 of proposition 3.3 (the point 1 will be proved below, F is a subset of F'). Let $y \in F'$ and $\mathcal{I} := \mathcal{I}(y, d^{-(\zeta''-\zeta')n})$. Since the intervals $\mathcal{I}_i$ defining $\mathcal{I}(\theta_n)$ are centered at $e_i = p_\kappa(c_i)$, the set $p_1^{-1}(\mathcal{I})$ does not intersect any point $c_i \in H^+ \cup E$. The same property holds for $p_2^{-1}(\mathcal{I})$ with the $c_i \in H^-$. This implies that $M_y\{d^{-\zeta'' n}\}$ does not intersect $\mathcal{V}_{\theta_n} \cap L$ for any $n \geq n_{z,w}(y)$, and yields the point 2.

For the point 1, it suffices to verify that $p_\kappa(\mathcal{V} \cap L)$ has zero Lebesgue measure. Let $\mathcal{W} := \mathcal{V} \cap L$. Since $(z,w) \in L$ and $(z,w) \notin \mathcal{V} = \cup_{i=0}^{\infty} f^i(\mathcal{C})$, the complex line L is not an algebraic subset of the hypersurface $f^i(\mathcal{C})$ for any $i \geq 0$. In particular, $\mathcal{W}_i := f^i(\mathcal{C}) \cap L$ is finite for every $i \geq 0$. Hence $\mathcal{W} = \cup_{i \geq 0} \mathcal{W}_i$ satisfies $\mathsf{Leb}(p_\kappa(\mathcal{W})) = 0$. We finally set $F := F' \setminus p_\kappa(\mathcal{W})$, that completes the proof of proposition 3.3.

3.4 Proof of theorem 3.2

We set $\epsilon := \frac{1}{2}(1 - (\theta + \zeta'')) > 0$ (see (2)). Let (z,w) be distinct points in $\mathbb{P}^k \setminus (\mathcal{D} \cup \mathcal{V})$ and consider some $y \in F$ provided by proposition 3.3: the meridian M_y does not intersect $\mathcal{V}$ and its neighbourhood $M_y\{d^{-\zeta'' n}\}$ in L does not intersect $\mathcal{V}_{\theta_n}$ for every $n \geq n_{z,w}(y)$.

We set $n_{z,w} := n_{z,w}(y)$ and denote $M := M_y$ for sake of simplicity. Let $\gamma : [0,1] \to L$ be the natural parametrization of M. We define $\Delta_n := M\{d^{-\zeta'' n}/2\}$ and $\tilde{\Delta}_n := M\{d^{-\zeta'' n}\}$. Let us apply Briend-Duval's theorem with $\eta = d^{-\theta n}$, $l = \theta_n$ and $\Delta_n \Subset$

$\tilde{\Delta}_n \subset L \setminus \mathcal{V}_{\theta_n}$. Since $n > \theta' n \geq \theta_n = l$ and $\tau_*/d^{\theta_n} < d^{-\theta n}$ (see (1)), there exist $(1-d^{-\theta n})d_t^n$ inverse branches on the disc Δ_n satisfying:

$$\mathsf{diam}\, g_n(\Delta_n) \leq \tilde{c}\, d^{-n/2} \left(d^{-\theta n}\, \mathsf{mod}\, \left[\tilde{\Delta}_n \setminus \Delta_n\right]\right)^{-1/2}. \tag{6}$$

It remains to bound the modulus of $A_n := \tilde{\Delta}_n \setminus \Delta_n$. Let Λ_n be the set of curves joining the boundary components of A_n. We denote by area_s and by l_s the area and the length in L with respect to the spherical metric s. The following estimates hold up to multiplicative constants. We have $l_\mathsf{s}(\lambda) \geq d^{-\zeta'' n}$ for any $\lambda \in \Lambda_n$, hence $l_\mathsf{s}(\Lambda_n) = \inf_{\lambda\in\Lambda_n} l_\mathsf{s}(\lambda) \geq d^{-\zeta'' n}$. The inequalities $\mathsf{area}_\mathsf{s}(A_n) \leq \mathsf{area}_\mathsf{s}(\tilde{\Delta}_n) \leq d^{-\zeta'' n}$ then imply:

$$\mathsf{mod}\,(A_n) = \sup_{\rho \in \mathsf{conf}\, A_n} \frac{l_\rho(\Lambda)^2}{\mathsf{area}_\rho(A_n)} \geq \frac{l_\mathsf{s}(\Lambda_n)^2}{\mathsf{area}_\mathsf{s}(A_n)} \geq \frac{d^{-2\zeta'' n}}{d^{-\zeta'' n}} = d^{-\zeta'' n}. \tag{7}$$

From (6), (7) and $\epsilon = \frac{1}{2}(1-(\theta+\zeta''))$, we deduce that $\mathsf{diam}\, g_n(\Delta_n) \leq \tilde{c}\, d^{-\epsilon n}$. That completes the proof of theorem 3.2.

3.5 Volume of neighbourhoods

This subsection is devoted to the proof of proposition 3.1: we want to show $\mathrm{Vol}\,(\mathcal{D}) = 0$, where $\mathcal{D} = \bigcap_{n \geq n_0} \bigcup_{p \geq n} \mathcal{V}_{\theta_p}[d^{-\zeta p}]$. We recall that $\mathcal{V}_{\theta_p}[d^{-\zeta p}]$ is the $d^{-\zeta p}$-neighbourhood of $\cup_{i=1}^{\theta_p} f^i(\mathcal{C})$ and that $\zeta > \theta'/2$. The proof is based on the following lemma (see [DS4], lemma 2.3.8).

Lemma 3.4 *Let $X \subset \mathbb{P}^k$ be an algebraic subvariety of dimension m and degree q. Then $\mathrm{Vol}\, X[\delta] \leq q\, \delta^{2(k-m)}$ for any $\delta > 0$, up to a multiplicative constant independent of X.*

We deduce $\mathrm{Vol}\,(\mathcal{D}) = 0$ as follows. We set $p \geq n \geq n_0$ and apply lemma 3.4 with $X = \mathcal{V}_{\theta_p}$ and $\delta = d^{-\zeta p}$ (here $k - m = 1$ and $q = \deg \mathcal{V}_{\theta_p} \leq \tau_{\theta_p}$). We obtain with $\tau_{\theta_p} \leq d^{\theta' p}$ (see (1)): $\mathrm{Vol}\, \mathcal{V}_{\theta_p}[d^{-\zeta p}] \leq \tau_{\theta_p} (d^{-\zeta p})^2 \leq d^{-(2\zeta - \theta')p}$. Hence:

$$\forall n \geq n_0 \ , \ \mathrm{Vol}\,(\mathcal{D}) \leq \mathrm{Vol} \bigcup_{p \geq n} \mathcal{V}_{\theta_p}[d^{-\zeta p}] \leq c_{2\zeta - \theta'}\, d^{-(2\zeta-\theta')n}.$$

This yields $\mathrm{Vol}\,(\mathcal{D}) = 0$ when n tends to infinity.

Proof of lemma 3.4: The argument is based on Lelong's inequality. Let $\mathcal{E}$ be a *maximal* δ-separated set in X for the ambient metric: this means that $d(a,b) \geq \delta$ for any pair of distinct elements of $\mathcal{E}$, and that for any $x \in X$ there exists $a \in \mathcal{E}$ satisfying $d(a,x) < \delta$. Since $X[\delta] \subset \cup_{a \in \mathcal{E}} B_a(2\delta)$, we get up to a multiplicative constant:

$$\mathrm{Vol}\, X[\delta] \leq (2\delta)^{2k}\, \mathsf{Card}\ \mathcal{E}. \tag{8}$$

We now give an upper bound for $\mathsf{Card}\,\mathcal{E}$. Observe that $\mathrm{Vol}\,X$ is equal to the degree of X, and that the balls $(B_a(\delta/2))_{a\in\mathcal{E}}$ are mutually disjoint. Thus:

$$q = \mathrm{Vol}\,X \geq \sum_{a\in\mathcal{E}} \mathrm{Vol}\,(X \cap B_a(\delta/2))\,.$$

Now Lelong's inequality asserts that $\mathrm{Vol}\,(X\cap B_a(\delta/2)) \geq \delta^{2m}$ for any $a\in\mathcal{E}$, up to a multiplicative constant. Hence $\mathsf{Card}\,\mathcal{E} \leq q\,\delta^{-2m}$, as desired. □

4 Proof of theorem A

We set $\mathcal{S} := \mathcal{V}\cup\mathcal{D}\cup f(\mathcal{D})\cup \mathsf{Per}(f)$, where $\mathcal{D}$ is defined in subsection 3.2. We have $\mathrm{Vol}\,(\mathcal{S}) = 0$ since $\mathrm{Vol}\,(\mathcal{D}) = 0$. Let us recall the statement of theorem A.

Theorem A: *Let $z\in\mathbb{P}^k\setminus\mathcal{S}$. There exist an s-invariant set $\Sigma'\subset\Sigma$ of full ν-measure and an f-invariant set $\mathcal{J}'\subset\mathcal{J}$ of full μ-measure satisfying the following properties. For any $\tilde{\alpha}\in\Sigma'$, the point $\omega(\tilde{\alpha}) := \lim_{n\to\infty} z_n(\tilde{\alpha})\in\mathcal{J}'$ is well defined. We have $\omega_*\nu = \mu$ and the following diagram commutes:*

$$\begin{array}{ccc} \Sigma' & \xrightarrow{\;s\;} & \Sigma' \\ {\scriptstyle\omega}\downarrow & & \downarrow{\scriptstyle\omega} \\ \mathcal{J}' & \xrightarrow{\;f\;} & \mathcal{J}' \end{array}$$

Moreover there exist $\theta,\epsilon>0$, $n_z\geq 1$ and $\tilde{n}:\Sigma'\to\mathbb{N}$ larger than n_z such that:

1. *$d(z_n(\tilde{\alpha}),\omega(\tilde{\alpha})) \leq \tilde{c}_\epsilon\, d^{-\epsilon n}$ for every $\tilde{\alpha}\in\Sigma'$ and $n\geq \tilde{n}(\tilde{\alpha})$,*
2. *$\nu(\{\tilde{n}\leq q\}) \geq 1 - c_\theta\, d^{-\theta q}$ for every $q\geq n_z$.*

We shall use theorem 3.2 and the method of coding trees introduced in [PUZ] for $(\mathbb{P}^1, f,\mu)$. We recall that $\mathcal{A} = \{1,\ldots,d_t\}$. Let $z\notin\mathcal{S}$ and $\{w_\alpha\,,\,\alpha\in\mathcal{A}\} := f^{-1}(z)$. By the very definition of $\mathcal{S}$, the cardinal of $f^{-1}(z)$ is equal to d_t and $w_\alpha\neq z$, $w_\alpha\notin\mathcal{V}\cup\mathcal{D}$ for every $\alpha\in\mathcal{A}$. We denote by L_α the projective line in $\mathbb{P}^k$ containing (z,w_α) and apply theorem 3.2: let γ_α be an injective smooth path joining (z,w_α) and $(\Delta_n(\alpha))_n\subset L_\alpha$ be a decreasing sequence of discs containing γ_α provided by that theorem. We set $n_z := \max\{n_{z,w_\alpha}\,,\,\alpha\in\mathcal{A}\}$.

Let us fix $\tilde{\alpha} = (\alpha_n)_{n\geq 0}\in\Sigma$. We define inductively injective smooth paths $\gamma_n(\tilde{\alpha}) : [0,1]\to\mathbb{P}^k\setminus\mathcal{V}$ and points $z_n(\tilde{\alpha})\in\mathbb{P}^k\setminus\mathcal{V}$. We first set $\gamma_0(\tilde{\alpha}) := \gamma_{\alpha_0}$. This path joins $z = \gamma_0(\tilde{\alpha})(0)$ and $w_{\alpha_0} = \gamma_0(\tilde{\alpha})(1) =: z_0(\tilde{\alpha})$. Assume that the paths $\gamma_j(\tilde{\alpha})$ and the points $z_j(\tilde{\alpha})$ have been defined for $0\leq j\leq n-1$. We let $\gamma_n(\tilde{\alpha})$ to be the lift of γ_{α_n} by f^n with starting point $\gamma_n(\tilde{\alpha})(0) = z_{n-1}(\tilde{\alpha})$. This path is well defined since γ_{α_n} does not intersect $\mathcal{V}$. We finally let $z_n(\tilde{\alpha}) := \gamma_n(\tilde{\alpha})(1)$.

We note that $z_{n-1}(\tilde\alpha)$ and $z_n(\tilde\alpha)$ are the endpoints of $\gamma_n(\tilde\alpha)$ and that $z_n(\Sigma) = f^{-(n+1)}(z)$ has cardinal d_t^{n+1}. The reader will easily check the relation $f \circ z_n(\tilde\alpha) = z_{n-1} \circ s(\tilde\alpha)$. Observe also that $\gamma_n(\tilde\alpha)$ and $z_n(\tilde\alpha)$ depend only on $\pi_n(\tilde\alpha) = (\alpha_0, \ldots, \alpha_n)$. The following lemma is a consequence of theorem 3.2 and the fact that $\gamma_\alpha[0,1] \subset \Delta_n(\alpha)$.

Lemma 4.1 *For every $\alpha \in \mathcal{A}$ and $n \geq n_z$, there exist at least $(1 - d^{-\theta n})d_t^n$ elements $(\alpha_0, \ldots, \alpha_{n-1}) \in \mathcal{A}^n$ such that $\mathsf{diam}\, \gamma_n(\alpha_0, \ldots, \alpha_{n-1}, \alpha) \leq \tilde c\, d^{-\epsilon n}$.*

Let $\Omega_n := \{\tilde\alpha \in \Sigma\, ,\, \mathsf{diam}\, \gamma_n(\tilde\alpha) > \tilde c\, d^{-\epsilon n}\}$ and $\mathcal{B}_n$ be the collection of $(n+1)$-cylinders $\{C_n(\tilde\alpha)\ ,\ \tilde\alpha \in \Omega_n\}$. We have $\Omega_n = \mathcal{B}_n^*$. Let us also define:

$$\Omega(n) \ := \ \bigcup_{p \geq n} \Omega_p \ = \ \bigcup_{p \geq n} \mathcal{B}_p^* \, .$$

Lemma 4.2 *For any $n \geq n_z$, we have:*

1. *$\mathsf{Card}\ (\mathcal{B}_n) \leq d_t^{n+1} d^{-\theta n}$.*
2. *$\nu(\Omega_n) \leq d^{-\theta n}$, hence $\nu(\Omega(n)) \leq c_\theta\, d^{-\theta n}$.*
3. *if $\tilde\alpha \notin \Omega(n)$, then $d(z_{m-1}(\tilde\alpha), z_m(\tilde\alpha)) \leq \tilde c\, d^{-\epsilon m}$ for any $m \geq n$.*

Proof: We have $\mathcal{B}_n = \{C_n(\tilde\alpha)\ ,\ \mathsf{diam}\, \gamma_n(\tilde\alpha) > \tilde c\, d^{-\epsilon n}\}$. For every $\alpha \in \mathcal{A}$, we set $\mathcal{B}_n(\alpha) \subset \mathcal{B}_n$ to be the collection of $(n+1)$-cylinders whose last coordinate is equal to α. The lemma 4.1 implies that $\mathsf{Card}\ (\mathcal{B}_n(\alpha)) \leq d_t^n d^{-\theta n}$ and thus $\mathsf{Card}\ (\mathcal{B}_n) = \sum_{\alpha \in \mathcal{A}} \mathsf{Card}\ (\mathcal{B}_n(\alpha)) \leq d_t^{n+1} d^{-\theta n}$, which is the point 1. The point 2 follows:

$$\nu(\Omega_n) = \nu(\mathcal{B}_n^*) = \mathsf{Card}\ (\mathcal{B}_n)/d_t^{n+1} \leq d^{-\theta n}.$$

For the point 3, observe that $d(z_{m-1}(\tilde\alpha), z_m(\tilde\alpha)) \leq \mathsf{diam}\, \gamma_m(\tilde\alpha)$. If $\tilde\alpha \notin \Omega(n)$, then $\tilde\alpha \notin \Omega_m$ for any $m \geq n$, hence $\mathsf{diam}\, \gamma_m(\tilde\alpha) \leq \tilde c\, d^{-\epsilon m}$. □

Let $\Omega := \bigcap_{n \geq n_z} \Omega(n) = \limsup_{n \geq n_z} \Omega_n$. The set $\Sigma'' := \Sigma \setminus \Omega$ has full ν-measure since $\nu(\Omega) \leq \nu(\Omega(n)) \leq c_\theta d^{-\theta n}$ for any $n \geq n_z$. For every $\tilde\alpha \in \Sigma''$, we define $\tilde n(\tilde\alpha)$ to be the least integer $n \geq n_z$ satisfying $\tilde\alpha \notin \Omega(n)$. Let $\Theta_q := \{\tilde n \leq q\}$.

Lemma 4.3

1. *$\omega(\tilde\alpha) = \lim_{n \to \infty} z_n(\tilde\alpha)$ is well defined for every $\tilde\alpha \in \Sigma''$.*
2. *$d(z_n(\tilde\alpha), \omega(\tilde\alpha)) \leq \tilde c_\epsilon\, d^{-\epsilon n}$ for every $n \geq \tilde n(\tilde\alpha)$.*
3. *$\omega : \Sigma'' \to \mathbb{P}^k$ satisfies $\omega_* \nu = \mu$.*
4. *$\nu(\Theta_q) \geq 1 - c_\theta\, d^{-\theta q}$ for any $q \geq n_z$.*

Proof: The points 1, 2 and 4 come from lemma 4.2(3,2) and the definition of $\tilde{n}(\tilde{\alpha})$. Now we prove the point 3. Let us consider the surjective map $z_n : \Sigma'' \to f^{-(n+1)}(z)$. Since $z_n(\tilde{\alpha})$ depends only on $\underline{\alpha} := (\alpha_0, \ldots, \alpha_n) \in \mathcal{A}^{n+1}$, the measure $z_{n*}\nu$ is equal to:

$$z_{n*}\nu = \sum_{\underline{\alpha} \in \mathcal{A}^{n+1}} \nu\left(\Sigma'' \cap C_n(\underline{\alpha})\right) \delta_{z_n(\underline{\alpha})} = \frac{1}{d_t^{n+1}} \sum_{f^{n+1}(y)=z} \delta_y = \mu_{n+1,z}.$$

Since $z \notin \mathcal{S}$ and $\mathcal{E} \subset \mathcal{V} \subset \mathcal{S}$, the sequence of probability measures $(\mu_{n,z})_n$ converges to μ (see subsection 2.1). Hence it remains to prove $z_{n*}\nu \to \omega_*\nu$, meaning that $\int_{\Sigma''} \varphi \circ z_n \, d\nu \to \int_{\Sigma''} \varphi \circ \omega \, d\nu$ for every test function $\varphi : \mathbb{P}^k \to \mathbb{R}$. But this follows from point 1 and Lebesgue convergence theorem. $\square$

It remains to define $\Sigma', \mathcal{J}'$ and to verify the relation $f \circ \omega = \omega \circ s$ on Σ'. The lemma 4.3(3) implies that $\Sigma_* := \omega(\Sigma'')$ satisfies $\mu(\Sigma_*) = \nu(\omega^{-1}\Sigma_*) \geq \nu(\Sigma'') = 1$. We define $\mathcal{J}' := \bigcap_{n \in \mathbb{Z}} f^n(\mathcal{J} \cap \Sigma_*)$ and $\Sigma' := \bigcap_{n \in \mathbb{Z}} s^n(\Sigma'' \cap \omega^{-1}\mathcal{J}')$. These are invariant subsets of full measure. We obtain $f \circ \omega = \omega \circ s$ on Σ' by taking limits in $f \circ z_n(\tilde{\alpha}) = z_{n-1} \circ s(\tilde{\alpha})$. That completes the proof of theorem A.

5 Proof of theorem B

Let us recall the statement.

Theorem B: *Let $\psi \in \mathcal{U}$ be a μ-centered observable and ω be a coding map provided by theorem A. Let $\chi := \psi \circ \omega$ and $1 \leq p < +\infty$.*

1. *there exist $\hat{c}_p, \lambda_p > 0$ such that $\| \chi - \mathbb{E}(\chi | \mathcal{C}_n) \|_p \leq \hat{c}_p \, e^{-n\lambda_p}$ for every $n \geq 0$.*
2. *$R_j(\chi) := \int_\Sigma \chi \cdot \chi \circ s^j \, d\nu$ satisfies $|R_j(\chi)| \leq 2 \| \chi \|_2 \, \hat{c}_2 \, e^{-(j-1)\lambda_2}$ for every $j \geq 1$.*

5.1 Proof of theorem B(1)

We set $\chi_B := \chi.1_B$ for any $B \subset \Sigma$ and use the following estimates provided by theorem A. We recall that $\Theta_n = \{\tilde{n}(\tilde{\alpha}) \leq n\}$.

($\star$) $d(z_n(\tilde{\alpha}), \omega(\tilde{\alpha})) \leq \tilde{c}_\epsilon \, d^{-\epsilon n}$ for every $\tilde{\alpha} \in \Theta_n$,

($\star\star$) $\nu(\Theta_n) \geq 1 - c_\theta \, d^{-n\theta}$ for every $n \geq n_z$.

We will need the following lemma, which is a direct consequence of ($\star$).

Lemma 5.1 *Let $\tilde{\alpha} \in \Theta_n$ and $\tilde{\beta} \in C_n(\tilde{\alpha}) \cap \Theta_n$. Then $d(\omega(\tilde{\alpha}), \omega(\tilde{\beta})) \leq 2 \, \tilde{c}_\epsilon \, d^{-\epsilon n}$.*

5.1.1 The Hölder case

Let ψ be an h-Hölder and μ-centered observable on $\mathbb{P}^k$. We set $\chi := \psi \circ \omega$. The theorem B(1) is a consequence of the following estimates, which hold for every $n \geq n_z$.

Lemma 5.2 $\left\| \chi_{\Theta_n^c} - \mathbb{E}(\chi_{\Theta_n^c}|\mathcal{C}_n) \right\|_p \leq 2 \| \chi \|_\infty \left(c_\theta \, d^{-n\theta} \right)^{1/p}$.

Proof: The left hand side is less than $2 \left\| \chi_{\Theta_n^c} \right\|_p$ by Jensen inequality. Then the conclusion follows from $(\star\star)$. $\square$

Lemma 5.3 $\| \chi_{\Theta_n} - \mathbb{E}(\chi_{\Theta_n}|\mathcal{C}_n) \|_p \leq c \, d^{-n\tau}$ *for some* $c, \tau > 0$.

Proof: We denote $\varphi := \chi_{\Theta_n} - \mathbb{E}(\chi_{\Theta_n}|\mathcal{C}_n)$ and estimate $\left\| \varphi_{\Theta_n^c} \right\|_p$, $\| \varphi_{\Theta_n} \|_p$. Since $\varphi_{\Theta_n^c} = -\mathbb{E}(\chi_{\Theta_n}|\mathcal{C}_n) \cdot 1_{\Theta_n^c}$, we have:

$$\left\| \varphi_{\Theta_n^c} \right\|_p \leq \| \mathbb{E}(\chi_{\Theta_n}|\mathcal{C}_n) \|_{2p} \cdot \nu(\Theta_n^c)^{1/2p} \leq \| \chi \|_{2p} \cdot \left(c_\theta \, d^{-n\theta} \right)^{1/2p}.$$

We now deal with $\| \varphi_{\Theta_n} \|_p$. For every $\tilde{\alpha} \in \Theta_n$, let $\nu_{\tilde{\alpha}}$ be the conditional measure of ν on the cylinder $C_n(\tilde{\alpha})$. We have for every $\tilde{\alpha} \in \Theta_n$:

$$\varphi_{\Theta_n}(\tilde{\alpha}) = \int_{C_n(\tilde{\alpha}) \cap \Theta_n} \left(\chi(\tilde{\alpha}) - \chi(\tilde{\beta}) \right) d\nu_{\tilde{\alpha}}(\tilde{\beta}) + \chi(\tilde{\alpha}) \cdot \nu_{\tilde{\alpha}}(C_n(\tilde{\alpha}) \cap \Theta_n^c). \tag{9}$$

We deduce from $\chi = \psi \circ \omega$, lemma 5.1 and the fact that ψ is h-Hölder:

$$\forall \tilde{\alpha} \in \Theta_n \ , \ |\varphi_{\Theta_n}(\tilde{\alpha})| \leq \left(2 \, \tilde{c}_\epsilon \, d^{-n\epsilon} \right)^h + \| \chi_{\Theta_n} \|_\infty \cdot \nu_{\tilde{\alpha}}(C_n(\tilde{\alpha}) \cap \Theta_n^c).$$

Hence we get for every $p \geq 1$ up to a multiplicative constant:

$$\forall \tilde{\alpha} \in \Theta_n \ , \ |\varphi_{\Theta_n}(\tilde{\alpha})|^p \leq d^{-nhp\epsilon} + \| \chi_{\Theta_n} \|_\infty^p \cdot \nu_{\tilde{\alpha}}(C_n(\tilde{\alpha}) \cap \Theta_n^c).$$

By integrating over Θ_n and using $(\star\star)$, we deduce:

$$\| \varphi_{\Theta_n} \|_p^p \leq d^{-nhp\epsilon} + \| \chi \|_\infty^p \cdot c_\theta d^{-n\theta}.$$

That completes the proof of the lemma. $\square$

5.1.2 The general case $\psi \in \mathcal{U}$

Let $\psi : \mathbb{P}^k \to \mathbb{R} \cup \{-\infty\}$ be a μ-centered observable in $\mathcal{U}$: the function e^ψ is h-Hölder and satisfies $\psi \geq \log d(\cdot, \mathcal{N}_\psi)^\rho$ on $\mathbb{P}^k$ (see definition 2.1). Observe in particular that ψ is bounded from above. We recall that $\mathcal{N}_\psi[r]$ is the r-neighbourhood of $\{\psi = -\infty\}$ and that $\chi = \psi \circ \omega$. We consider the following subsets of Σ:

$$\mathcal{S}_n := \Theta_n^c \setminus \mathcal{N}_n \quad , \quad \Gamma_n = \Theta_n \setminus \mathcal{N}_n \quad , \quad \mathcal{N}_n := \omega^{-1} \left(\mathcal{N}_\psi[d^{-n(h\epsilon/2\rho)}] \right).$$

We shall need the following observations. First, we have $\nu(\mathcal{N}_n) = \mu(\mathcal{N}_\psi[d^{-n(h\epsilon/2\rho)}]) \leq d^{-n\gamma(h\epsilon/2\rho)}$ up to a multiplicative constant (see subsection 2.1). We deduce from $(\star\star)$:

$$\nu(\Gamma_n^c) = \nu(\Theta_n^c \cup \mathcal{N}_n) \leq c_\theta d^{-n\theta} + d^{-n\gamma(h\epsilon/2\rho)} \leq c\, d^{-n\eta} \tag{10}$$

for some $c, \eta > 0$. Second, for every $\tilde{\alpha} \in \mathcal{N}_n^c = \mathcal{S}_n \cup \Gamma_n$, we have $\chi(\tilde{\alpha}) \geq \log d(\omega(\tilde{\alpha}), \mathcal{N}_\psi)^\rho \geq \log d^{-\rho n(h\epsilon/2\rho)}$, hence:

$$\|\, \chi_{\mathcal{S}_n \cup \Gamma_n} \|_\infty \leq n\, (h\epsilon \log d)/2. \tag{11}$$

The theorem B(1) is now a consequence of the following estimates.

Lemma 5.4 $\|\, \chi_{\mathcal{N}_n} - \mathbb{E}(\chi_{\mathcal{N}_n}|\mathcal{C}_n) \|_p \leq \left(\kappa\, d^{-n(h\epsilon/2\rho)\cdot(\gamma/2)}\right)^{1/p}$.

Proof: The left hand side is less than $2 \|\, \chi_{\mathcal{N}_n} \|_p$. Proposition 2.2 yields $\|\, \chi_{\mathcal{N}_n} \|_p = \|\, \psi \circ \omega \cdot 1_{\mathcal{N}_n} \|_p \leq \left(\kappa\, d^{-n(h\epsilon/2\rho)\cdot(\gamma/2)}\right)^{1/p}$ for every n such that $d^{-n(h\epsilon/2\rho)} < 1/2$. $\square$

Lemma 5.5 $\|\, \chi_{\mathcal{S}_n} - \mathbb{E}(\chi_{\mathcal{S}_n}|\mathcal{C}_n) \|_p \leq n\, (h\epsilon \log d) \cdot (c\, d^{-n\eta})^{1/p}$.

Proof: The left hand side is less than $2 \|\, \chi_{\mathcal{S}_n} \|_p$. We conclude by using (10) and (11) (observe that $\mathcal{S}_n \subset \Gamma_n^c$). $\square$

Lemma 5.6 $\|\, \chi_{\Gamma_n} - \mathbb{E}(\chi_{\Gamma_n}|\mathcal{C}_n) \|_p \leq c\, d^{-n\tau}$ *for some* $c, \tau > 0$.

Proof: We follow the proof of lemma 5.3: we set $\varphi := \chi_{\Gamma_n} - \mathbb{E}(\chi_{\Gamma_n}|\mathcal{C}_n)$ and estimate $\left\|\, \varphi_{\Gamma_n^c} \right\|_p$, $\|\, \varphi_{\Gamma_n} \|_p$. The line (10) yields:

$$\left\|\, \varphi_{\Gamma_n^c} \right\|_p \leq \|\, \mathbb{E}(\chi_{\Gamma_n}|\mathcal{C}_n) \|_{2p} \cdot \nu(\Gamma_n^c)^{1/2p} \leq \|\, \chi \|_{2p} \cdot \left(c\, d^{-n\eta}\right)^{1/2p}.$$

Now we deal with $\|\, \varphi_{\Gamma_n} \|_p$. We can write as in (9):

$$\forall \tilde{\alpha} \in \Gamma_n\ ,\ \varphi(\tilde{\alpha}) = \int_{C_n(\tilde{\alpha}) \cap \Gamma_n} \left(\chi(\tilde{\alpha}) - \chi(\tilde{\beta})\right) d\nu_{\tilde{\alpha}}(\tilde{\beta}) + \chi(\tilde{\alpha}) \cdot \nu_{\tilde{\alpha}}(C_n(\tilde{\alpha}) \cap \Gamma_n^c). \tag{12}$$

Let $\tilde{\alpha} \in \Gamma_n$ and $\tilde{\beta} \in C_n(\tilde{\alpha}) \cap \Gamma_n$. We deduce from $(\tilde{\alpha}, \tilde{\beta}) \notin \mathcal{N}_n$ that $e^\psi \circ \omega(\tilde{\alpha})$ and $e^\psi \circ \omega(\tilde{\beta})$ are larger than $d^{-nh\epsilon/2}$. This implies:

$$|\chi(\tilde{\alpha}) - \chi(\tilde{\beta})| = |\log e^\psi \circ \omega(\tilde{\alpha}) - \log e^\psi \circ \omega(\tilde{\beta})| \leq d^{nh\epsilon/2} |e^\psi \circ \omega(\tilde{\alpha}) - e^\psi \circ \omega(\tilde{\beta})|.$$

Using lemma 5.1 and the fact that e^ψ is h-Hölder, the last term is less than $d^{nh\epsilon/2} \cdot (2\, \tilde{c}_\epsilon\, d^{-n\epsilon})^h$. Then we deduce from (12), up to a multiplicative constant:

$$\forall \tilde{\alpha} \in \Gamma_n\ ,\ |\varphi(\tilde{\alpha})| \leq d^{-nh\epsilon/2} + \|\, \chi_{\Gamma_n} \|_\infty \cdot \nu_{\tilde{\alpha}}(C_n(\tilde{\alpha}) \cap \Gamma_n^c).$$

Taking the p-th power, integrating over Γ_n and using (10), (11), we obtain up to a multiplicative constant:

$$\|\, \varphi_{\Gamma_n} \|_p^p \leq d^{-nhp\epsilon/2} + (n(h\epsilon \log d)/2)^p \cdot c\, d^{-n\eta}.$$

That completes the proof of the lemma. $\square$

5.2 Proof of theorem B(2)

Let $\psi \in \mathcal{U}$ be a μ-centered observable and $\chi = \psi \circ \omega$. Let $j \geq 1$ and $n \geq 0$ to be specified later. We set $\chi_n := \mathbb{E}(\chi|\mathcal{C}_n)$ and write:

$$\chi \cdot \chi \circ s^j = (\chi - \chi_n) \cdot \chi \circ s^j + \chi_n \cdot (\chi \circ s^j - \chi_n \circ s^j) + \chi_n \cdot \chi_n \circ s^j.$$

By using the s-invariance of ν and Jensen inequality $\| \chi_n \|_2 \leq \| \chi \|_2$, we deduce:

$$|R_j(\chi)| = \left| \int_\Sigma \chi \cdot \chi \circ s^j \, d\nu \right| \leq 2 \| \chi \|_2 \| \chi - \chi_n \|_2 + \left| \int_\Sigma \chi_n \cdot \chi_n \circ s^j \, d\nu \right|. \tag{13}$$

The variables χ_n and $\chi_n \circ s^j$ respectively depend on $(\xi_0, \ldots, \xi_n)$ and $(\xi_j, \ldots, \xi_{n+j})$, where $\xi_n : \Sigma \to \mathcal{A}$ is the projection $\xi_n(\tilde{\alpha}) = \alpha_n$. These are independent variables when $n = j-1$, hence $\int_\Sigma \chi_n \cdot \chi_n \circ s^j \, d\nu = \int_\Sigma \chi_n \, d\nu \int_\Sigma \chi_n \circ s^j \, d\nu$ in that case. But this product is zero since χ is ν-centered. The conclusion then follows from (13) with $n = j-1$ and theorem B(1).

6 Proof of theorem C

Let us recall the statement.

Theorem C: *For every μ-centered observable $\psi \in \mathcal{U}$, we have:*

1. $\sigma := \lim_{n\to\infty} \frac{1}{\sqrt{n}} \| S_n(\psi) \|_2$ *exists, and* $\sigma^2 = \int_{\mathbb{P}^k} \psi^2 \, d\mu + 2 \sum_{j\geq 1} \int_{\mathbb{P}^k} \psi \cdot \psi \circ f^j \, d\mu$.
2. *If* $\sigma = 0$*, then* $\psi = u - u \circ f$ *μ-a.e. for some* $u \in L^2(\mu)$.
3. *If* $\sigma > 0$*, then* ψ *satisfies the* σ*-ASIP.*

The points 1 and 2 are consequences of classical lemma 6.1 below, whose condition $\sum_{j\geq 1} j |R_j(\varphi)| < \infty$ is fulfilled by theorem B(2). The point 3 follows from proposition 2.2, theorem B(1) and Philipp-Stout's theorem (see subsection 2.4).

Lemma 6.1 *Let (X, g, m) be a dynamical system and $\varphi \in L^2(m)$ be a m-centered observable. We denote $S_n(\varphi) = \sum_{j=0}^{n-1} \varphi \circ g^j$ and $R_j(\varphi) = \int_X \varphi \cdot \varphi \circ g^j \, dm$. Let $\sigma^2 := R_0(\varphi) + 2 \sum_{j\geq 1} R_j(\varphi)$. If $\sum_{j\geq 1} j|R_j(\varphi)| < \infty$, then σ^2 is finite and we have:*

1. $\| S_n(\varphi) \|_2^2 = n\sigma^2 + O(1)$. *In particular,* $\lim_{n\to\infty} \frac{1}{n} \| S_n(\varphi) \|_2^2 = \sigma^2$.
2. $\sigma^2 = 0$ *if and only if* $\varphi = u - u \circ g$ *m-a.e. for some* $u \in L^2(m)$.

Proof: Let $S_n := S_n(\varphi)$ and $R_j := R_j(\varphi)$. Since m is g-invariant, we have $\| S_n \|_2^2 = nR_0 + 2 \sum_{j=1}^{n-1} (n-j) R_j$. We deduce for every $n \geq 1$:

$$\| S_n \|_2^2 = n \left(R_0 + 2 \sum_{j=1}^{\infty} R_j \right) + (-2) \left(\sum_{j=1}^{n-1} jR_j + \sum_{j=n}^{\infty} nR_j \right) = n\sigma^2 + A_n, \tag{14}$$

where $|A_n| \leq 2\sum_{j\geq 1} j|R_j|$. That proves the point 1. Let us show the point 2. Suppose $\sigma^2 = 0$. In view of (14), the function $u_p := \frac{1}{p}\sum_{n=1}^{p} S_n$ satisfies $\| u_p \|_2 \leq (2\sum_{j\geq 1} j|R_j|)^{1/2}$ for every $p \geq 1$. Let $u := \lim_{j\to\infty} u_{p_j}$ be a weak cluster point in $L^2(m)$ and observe that:

$$\forall j \geq 1 \ , \ u_{p_j} - u_{p_j} \circ g = \frac{1}{p_j}\sum_{n=0}^{p_j-1}(\varphi - \varphi \circ g^n) = \varphi - \frac{1}{p_j} S_{p_j}.$$

We deduce $\varphi = u - u \circ g$ m-a.e. by taking limits in $L^2(m)$: $\lim_{j\to\infty} u_{p_j} \circ g = u \circ g$ since m is g-invariant, and $\lim_{j\to\infty} \frac{1}{p_j} S_{p_j} = \int_X \varphi \, dm = 0$ by Von Neumann theorem. The reverse implication of the point 2 comes from $\sigma^2 = \lim_{n\to\infty} \frac{1}{n} \| S_n(\varphi) \|_2^2 = \lim_{n\to\infty} \frac{1}{n} \| u - u \circ g^n \|_2^2 = 0$. □

7 Proof of theorem D

We recall that $J := \log \mathsf{Jac}\, f - \int_{\mathbb{P}^k} \log \mathsf{Jac}\, f \, d\mu$, this is an unbounded μ-centered observable in $\mathcal{U}$. We set $\sigma_J := \lim_n \frac{1}{\sqrt{n}} \| S_n(J) \|_2$, which is well defined by theorem C. We denote by $\chi_1 \leq \ldots \leq \chi_k$ the Lyapunov exponent of μ, they are larger than or equal to $\log d^{1/2}$.

Theorem D: *If the Lyapunov exponents of μ are minimal equal to $\log d^{1/2}$, then $\sigma_J = 0$ and μ is absolutely continuous with respect to Lebesgue measure.*

The first part $\sigma_J = 0$ will be proved in subsection 7.2. The second part is a consequence of theorem 7.1 below (that theorem will be proved in subsection 7.3 by using $\sigma_J = 0$). In the sequel, the maps f^n and $d_x f^n$ are implicitely written in some fixed charts of $\mathbb{P}^k$.

Theorem 7.1 *Assume that the Lyapunov exponents are minimal. Then for μ almost every $x \in \mathbb{P}^k$, there exists $\rho(x) > 0$ and a subsequence $(n_j(x))_{j\geq 1}$ such that $f^{n_j} \circ \left(x + d^{-n_j/2} \cdot \mathrm{Id}_{\mathbb{C}^k}\right) : B(\rho(x)) \to \mathbb{P}^k$ is injective.*

Proof of the second part of theorem D (abolute continuity): We use the notations of theorem 7.1. Let $x \in \mathbb{P}^k$ be a μ-generic point and set $n_j := n_j(x)$. Since f^{n_j} is injective on the ball $B_j := B_x(\rho(x) d^{-n_j/2})$ and μ has constant jacobian d^k (see subsection 2.1), we obtain $\mu(B_j) = \mu(f^{n_j}(B_j)) d^{-kn_j}$. Observe also that $\mathsf{Leb}(B_j) = \rho(x)^{2k} \left(d^{-n_j/2}\right)^{2k} = \rho(x)^{2k} d^{-kn_j}$ up to a multiplicative constant. We obtain therefore for μ-a.e. $x \in \mathbb{P}^k$:

$$\liminf_{r\to 0} \frac{\mu(B_x(r))}{\mathsf{Leb}(B_x(r))} \leq \liminf_{j\to\infty} \frac{\mu(B_j)}{\mathsf{Leb}(B_j)} = \liminf_{j\to\infty} \frac{\mu(f^{n_j}(B_j))}{\rho(x)^{2k}} \leq \frac{1}{\rho(x)^{2k}} < \infty.$$

That proves the absolute continuity of μ (see [Mat], theorem 2.12). □

7.1 Preliminaries

Observe that $J = \log \mathsf{Jac}\, f - \log d^k$ when the Lyapunov exponents are equal to $\log d^{1/2}$. Since the jacobian is a multiplicative function, we have in that case:

$$S_n(J) = \sum_{i=0}^{n-1} J \circ f^i(x) = \log \mathsf{Jac}\, f^n - \log d^{kn}. \tag{15}$$

The *singular values* $\delta_1 \leq \ldots \leq \delta_k$ of the linear map $A := d_x f^n$ are defined as the eigenvalues of $\sqrt{AA^*}$. In particular, there exist unitary matrices (U, V) such that $d_x f^n = U\, \mathsf{Diag}(\delta_1, \ldots, \delta_k)\, V$. We have therefore:

$$\delta_1 = \left\| (d_x f^n)^{-1} \right\|^{-1} \quad \text{and} \quad \prod_{i=1}^{k} {\delta_i}^2 = \mathsf{Jac}\, f^n(x) \geq \delta_1^{2k}. \tag{16}$$

For any $\rho, \tau > 0$ and $n \geq 1$, we define:

$$\mathcal{B}_n(\rho) := \left\{ x \in \mathbb{P}^k \,,\, f^n \circ (x + d_x f^n)^{-1} : B(\rho) \to \mathbb{P}^k \text{ is an injective map} \right\},$$

$$\mathcal{R}_n(\tau) := \left\{ x \in \mathbb{P}^k \,,\, \left\| (d_x f^n)^{-1} \right\|^{-1} \geq d^{n/2}/\tau \right\}.$$

The following estimates were proved by Berteloot-Dupont [BeDu]. They hold for every system $(\mathbb{P}^k, f, \mu)$ whose Lyapunov exponents satisfy $\chi_k < 2\chi_1$.

Theorem 7.2 *There exists* $\alpha :]0,1] \to \mathbb{R}_+^*$ *satisfying* $\lim_{\rho \to 0} \alpha(\rho) = 1$ *and for* $n \geq 1$*:*

1. $\mu(\mathcal{B}_n(\rho)) \geq \alpha(\rho)$,
2. $\mu(\mathcal{B}_n(\rho) \cap \mathcal{R}_n(\tau)^c) \leq (\rho\, \tau)^{-2}$.

That result implies the following lemma.

Lemma 7.3 *Let* $\rho \in]0,1]$*. There exists* $\mathcal{H} \subset \mathbb{P}^k$ *satisfying* $\mu(\mathcal{H}) = 1$ *and:*

$$\forall x \in \mathcal{H} \ , \ \exists n(x) \geq 1 \ , \ \forall n \geq n(x) \ , \ x \notin \mathcal{B}_n(\rho) \ \textit{or} \ \mathsf{Jac}\, f^n(x) \geq d^{kn}/n^{2k}.$$

PROOF: We apply proposition 7.2(2) with $\tau = n$ to get $\mu(\mathcal{B}_n(\rho) \cap \mathcal{R}_n(n)^c) \leq (\rho\, n)^{-2}$. Since $\sum_{n \geq 1} \mu(\mathcal{B}_n(\rho) \cap \mathcal{R}_n(n)^c) < \infty$, there exists by Borel-Cantelli lemma a subset $\mathcal{H}$ of full μ-measure satisfying:

$$\forall x \in \mathcal{H} \ , \ \exists n(x) \geq 1 \ , \ \forall n \geq n(x) \ , \ x \notin \mathcal{B}_n(\rho) \text{ or } x \in \mathcal{R}_n(n).$$

But $x \in \mathcal{R}_n(n)$ implies by (16): $\mathsf{Jac}\, f^n(x) \geq \left(d^{n/2}/n\right)^{2k} = d^{kn}/n^{2k}$. $\square$

7.2 Proof of the first part of theorem D ($\sigma_J = 0$)

Suppose that the exponents are minimal and that $\sigma_J = \lim_n \frac{1}{\sqrt{n}} \| S_n(J) \|_2 > 0$. Then J satisfies the CLT: if $V := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-1} e^{-u^2/2}\, du$, we get $\mu\left(\mathcal{G}_n := \{\frac{S_n(J)}{\sqrt{n}} \leq -\sigma_J\}\right) \geq V/2$ for n larger than some N (see subsection 2.4).

Let $\rho > 0$ be such that $\mu(\mathcal{B}_n(\rho)) > 1 - V/4$ for every $n \geq 1$. If we set $\mathcal{F}_n := \mathcal{B}_n(\rho) \cap \mathcal{G}_n$, then $\mathcal{F} := \limsup_{n \geq N} \mathcal{F}_n$ satisfies $\mu(\mathcal{F}) \geq V/4$. Let $x \in \mathcal{F} \cap \mathcal{H}$, where $\mathcal{H}$ is provided by lemma 7.3. Let $(n_j(x))_{j \geq 1}$ be such that $x \in \mathcal{F}_{n_j}$ for every $j \geq 1$. The inclusion $\mathcal{F}_{n_j} \subset \mathcal{G}_{n_j}$ yields $S_{n_j}(J)(x) \leq -\sigma_J \sqrt{n_j}$ for every $j \geq 1$. Since $S_{n_j}(J) = \log \mathsf{Jac}\, f^{n_j} - \log d^{kn_j}$ (the exponents are indeed minimal, see (15)), we deduce:

$$\forall j \geq 1 \ , \ \mathsf{Jac}\, f^{n_j}(x) \leq d^{kn_j} e^{-\sigma_J \sqrt{n_j}}. \tag{17}$$

But $\mathsf{Jac}\, f^{n_j}(x) \geq d^{kn_j}/n_j^{2k}$ for every $n_j \geq n(x)$, following from $x \in \mathcal{B}_{n_j}(\rho) \cap \mathcal{H}$ and lemma 7.3. That contradicts (17) when j tends to infinity.

7.3 Proof of theorem 7.1

We proved in subsection 7.2 that $\sigma_J = 0$. Hence $J = u - u \circ f$ μ-a.e. for some $u \in L^2(\mu)$ by theorem C. We obtain therefore:

$$u - u \circ f^n(x) = \sum_{i=0}^{n-1} J \circ f^i(x) = \log \mathsf{Jac}\, f^n(x) - \log d^{kn}. \tag{18}$$

Let $\epsilon > 0$ and $m \geq 1$ such that $\mathcal{M} := \{|u| \leq \log m\}$ satisfies $\mu(\mathcal{M}) \geq (1-\epsilon)^{1/2}$. Since μ is mixing, $\mathcal{M}_n := \mathcal{M} \cap f^{-n}\mathcal{M}$ satisfies $\mu(\mathcal{M}_n) \geq \mu(\mathcal{M})^2 - \epsilon \geq 1 - 2\epsilon$ for n larger than some N'. Let ρ be small and τ be large enough such that $\mu(\mathcal{B}_n(\rho) \cap \mathcal{R}_n(\tau)) \geq 1 - 2\epsilon$ for every $n \geq 1$. We define $\mathcal{T}_n := \mathcal{B}_n(\rho) \cap \mathcal{R}_n(\tau) \cap \mathcal{M}_n$ and $\mathcal{T} := \limsup_{n \geq N'} \mathcal{T}_n$. Observe that $\mu(\mathcal{T}) \geq 1 - 4\epsilon$. Let $x \in \mathcal{T}$ and $(n_j)_j$ (depending on x) such that $x \in \mathcal{T}_{n_j}$ for every $j \geq 1$. Since $x \in \mathcal{T}_{n_j} \subset \mathcal{B}_{n_j}(\rho)$, the map $f^{n_j} \circ (x + (d_x f^{n_j})^{-1}) : B(\rho) \to \mathbb{P}^k$ is injective.

Let $\Lambda_n = d^{-n/2} \cdot \mathrm{Id}_{\mathbb{C}^k}$. It is enough to prove that $d_x f^{n_j} = (U_j\, P_j\, V_j)\, \Lambda_{n_j}^{-1}$, where (U_j, V_j) are unitary matrices and P_j is a diagonal matrix with entries in $[a, b] \subset \mathbb{R}_+^*$ ((a, b) being independent of j). Indeed, this implies that $f^{n_j} \circ (x + \Lambda_{n_j})$ is injective on $B(\rho/b)$, completing the proof of theorem 7.1. We shall omit the subscript j for simplification, and denote by $\delta_1 \leq \ldots \leq \delta_k$ the singular values of $d_x f^n$. Let (U, V) be unitary matrices such that $d_x f^n = U\, \mathsf{Diag}(\delta_1, \ldots, \delta_k)\, V$ (see subsection 7.1). The fact that $x \in \mathcal{R}_n(\tau)$ yields:

$$\delta_1 = \left\| (d_x f^n)^{-1} \right\|^{-1} \geq d^{n/2}/\tau. \tag{19}$$

Now we give an upper bound for δ_k. Since $x \in \mathcal{T}_n \subset \mathcal{M}_n$, we have $(x, f^n(x)) \in \mathcal{M} = \{|u| \leq \log m\}$. This implies by (18):

$$d^{kn/2}/m \ \leq \ \prod_{i=1}^{k} \delta_i = \mathsf{Jac}\, f^n(x)^{1/2} \ \leq \ d^{kn/2} m.$$

We deduce from (19):

$$\delta_k \le \frac{\delta_1 \ldots \delta_{k-1}}{\delta_1^{k-1}} \delta_k = \frac{\mathsf{Jac}\, f^n(x)^{1/2}}{\delta_1^{k-1}} \le \frac{d^{kn/2} m}{(d^{n/2}/\tau)^{k-1}} = d^{n/2} \tau^{k-1} m.$$

Thus $\mathsf{Diag}(\delta_1, \ldots, \delta_k) = \Lambda_n^{-1} P$, where P is diagonal with entries in $[1/\tau, \tau^{k-1} m]$. We obtain finally $d_x f^n = U\, \Lambda_n^{-1}\, P\, V = (U\, P\, V)\, \Lambda_n^{-1}$, as desired.

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